Double Conformal Mapping: A Finite Mathematics to Model an Infinite World

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Abstract. The conformal model of Geometric Algebra suggests an intimate connection between mathematics and perception, in particular in the handling of the problem of infinity. The observed properties of phenomenal perspective suggest an extension to Hestenes’ conformal mapping by adding a second conformal stage that maps the infinity of external reality to a finite double conformal map.

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1. The Ontology of Mathematics

The history of mathematics, and particularly of algebra, has been a history of an incremental discovery of an apparently pre-existing structure. Concepts like zero, negative numbers, imaginary and complex numbers that seemed initially like inventions, or “tricks” contrived to simplify practical calculations, turn out in retrospect to have been more like discoveries of the hidden structure of mathematics. In retrospect it seems that there is one mathematical Truth, and new mathematical contrivances are only valid if they are consistent with that pre-existing Truth. It is as if negative and imaginary numbers had been there all along just waiting to be discovered as soon as the problems that they resolve were first encountered.

The discovery of Geometric Algebra [1] as the lynchpin that unites so many diverse specialized fields of mathematics under a single formalism, and its extraordinary extensions into first projective geometry, and then conformal geometry, has given us perhaps the first glimpses of the core of mathematics, allowing us to finally identify what it is that we have been unearthing

A version of this paper with color figures and animations is available at:
https://doubleconformal.wordpress.com/.
during the long history of mathematical discovery, and to begin to address the core question that it raises: What is this magnificent edifice that we have unearthed? What is mathematics?

If mathematics is indeed a pre-existing structure that is progressively being exposed with each new advance, where is that structure and what is its ontology? Many mathematicians apparently still hold to the Platonic ideal whereby mathematics and its laws are believed to have objective existence in an orthogonal dimension inaccessible to scientific scrutiny. We can “see” mathematical concepts with our minds eye, and record them on the page as figures or equations, but there is no way to “photograph” them or make an objective record of the mathematical Truth, because it supposedly has no material existence in the physical world known to science.

But if mathematical concepts are indeed in principle beyond scientific observation, this Platonic hypothesis is unfalsifiable, because its prediction, that mathematics cannot be observed, is identical to the null prediction that mathematics does not have independent existence, and thus the Platonic ideal is not a scientific hypothesis that can ever be confirmed nor refuted, but more of a belief, for those who are so inclined, to account for the apparent pre-existence of mathematical Truth. But that is not the only possible explanation. Lakoff and Nuñez [2] argue for a biological origin for mathematics: That mathematics is an artifact of how our brain makes sense of spatial reality, and that therefore mathematics and its laws will one day be identified as a physical process, or computational principle taking place within the physical mechanism of the brain. The apparent pre-existence of math is thus accounted for by the explanation that mathematical Truth is somehow hard-wired in the human mind and brain.

If the biological theory of mathematics is right, that changes fundamentally the entire program of mathematics, the object of our study, and why it is interesting or significant. Once we understand that by studying math, we are studying the computational principles of our own mind and brain, that opens the way for the next step to try and discover the physical principle behind biological mathematics in the brain, on the presumption that mind does not inhabit an orthogonal dimension inaccessible to scientific scrutiny, but rather, that mind is a physical process taking place in the physical mechanism of the brain. That process must leave artifacts in our experience of the world that reveal its underlying computational principles. I propose that the entire edifice of mathematics is exactly one such artifact.

This in turn changes our understanding of what mathematics must remain true to. It is generally assumed that the “right” mathematics is right exactly because it is true to the external reality that it is used to model. As with perception, the capacity to accurately and succinctly model physical reality is what makes mathematics useful. The biological mathematical hypothesis on the other hand suggests that what mathematics must remain true to is not objective external reality, but our internal mental model of it as it is constructed by our brain. Of course the pragmatic constraints of
natural selection would also generally require that perception be veridical, i.e. that our internal model of the world conforms to external reality at least close enough so we don't bump into walls or fall down stairs, and that is what keeps perception, and mathematics honest, or true to reality. There is however one aspect of external reality that is impossible in principle to capture explicitly in an internal model, and that is its practically infinite extent. We cannot perceive anything as farther than the dome of the night sky, because anything beyond a certain distance appears perceptually as if pasted on the dome of the sky. Size and distance are limited by the capacity of human perception. We see a parallel problem in mathematics, where the concept of infinity appears as a special paradoxical exception to the rest of the number line in that infinity is not a number, it is invalid to include it in mathematical terms. For example the term “infinity - 1” is meaningless, and division by zero is strictly verboten in mathematics because its result is infinity. Indeed anthropological studies with native Amazonian indians [3] suggest that the natural intuitive concept of number for people unschooled in mathematics exhibits this kind of compressed scale.

David Hestenes' [1] extensions to Geometric Algebra, the projective geometry, and the conformal geometry, clearly demonstrate how mathematics can be defined to handle infinities by a projection back to “finity”, a mapping from an infinite to a finite bounded range, in a manner that is highly suggestive of the way the issue is handled in perception. This in turn suggests a reversal in the usual hierarchy of mathematical concepts whereby projective and conformal geometries are seen as extensions to regular Euclidean geometry, and are expressed in terms of Euclidean geometry or regular algebra. If mathematics evolved from perception, as suggested by the biological theory of mathematics, then perhaps conformal geometry is the original primal form of mathematics, as it is expressed most directly in the human brain, and Euclidean geometry, and the conventional treatment of infinity as an anomaly on the number line, are derivative of a more primal conformal geometry wherein infinity is not a paradoxical or impossible concept, but is just another number at the end of the number line.

For example mathematics could just as well have been defined from the outset based on a stereographic projection of the conventional number line as shown in Figure 1 A, in which the integer divisions on the number line are projected onto the perimeter of the unit circle from a point at its zenith. Figure 1 B shows the unit circle “unwrapped” from a circle to a line of finite length, on which the integer divisions become ever more compressed as they approach “infinity”, with the number line terminating abruptly at a point called “infinity” in both the positive and negative directions. If the scale of numerical value were to shrink into the distance along with the compressed integer scale, each integer subdivision would still maintain its “true” integer value on the compressed scale, besides being presented at smaller and smaller scale with approach toward infinity. Since all algebraic relations between numerical values would also be preserved despite the compression, algebra defined on
Figure 1. A: A stereographic projection from the zenith of the unit circle projects the integers onto the circular circumference. B: The circle is “unwrapped” to a finite line whose endpoints represent positive and negative “infinity”.

this stereographic scale would be mathematically isomorphic to that on the conventional number line. It would, in effect, be a dualized representation that could be seen from two alternative perspectives: either from the warped perspective where it is functionally identical to the conventional number line (with the exception of the points at “infinity”), or from the actual distance in the representation in which the integer divisions get closer and closer as they approach “infinity”. The only difference between this warped number line and the conventional one occurs at “infinity” itself, which can be easily depicted and labeled on the warped number line, but must remain forever off the edge of the page in the conventional number line no matter how small the figure is scaled. Another difference is that on the warped scale “infinity” is no longer a paradoxical anomaly, but is just a regular number that can be used in equations, and division by zero would no longer be prohibited, it would simply be equal to “infinity”.

The significance of this insight is that conventional mathematics is impossible to implement as an analog, or analogical computation in a real physical system due to the problem of infinity, whereas a conformally warped mathematics is indeed possible to implement in an analogical mechanism that computes its mathematical results by direct physical analogy. This finding is extremely useful if we propose study mathematics not as a model of physical reality, but as a model of the computational mechanism in the brain that models reality, because it offers the first hints as to how we can devise an analogical mechanism that performs mathematical functions by the same principles as the human mind and brain.
2. Biological Mathematics

The Biological Theory of mathematics suggests that mathematics evolved from basic perception, and thus it inherits its most basic operational principles from the principles of perception. When I first encountered David Hestenes’ conformal projection in Geometric Algebra, I recognized a striking similarity to another kind of conformal projection I had been studying in visual perception, i.e. how we perceive the larger space around us, depicted in Figure 2 A. If you stand on a long straight road, or railway track, the sides of the road converge by perspective until they meet at a point on the horizon, the “vanishing point” at “perceptual infinity”. And if you turn around and look behind you, they meet at another point in the opposite direction too. And yet the sides of the road appear straight and parallel and equidistant throughout their length, even where they meet on the horizon! This is directly analogous to the stereographically warped number line discussed above. The most remarkable thing about this peculiar warp in perceptual space is the fact that most people never even notice it, and some even deny vociferously that they see anything shrunken smaller by perspective at all.

In case you have never noticed this warp in phenomenal perspective, and might question its truth, I conducted an experiment in a hallway at the Schepens Eye Research Institute [5] to demonstrate phenomenal perspective. Subjects placed in the hallway were given three cardboard models, as shown in Figure 3 A, and asked to choose which model most closely resembled their experience of the surrounding hallway: Not the shape that they “knew” it to have, but the shape that they experienced the hallway to have. Did it appear like a rectangular prism, like model A? Or did it look like a flat two-dimensional perspective projection as in model C? Or did the hallway appear
intermediate, a kind of bas-relief shape as in model B? Subjects responses were split about evenly between models A and B. Then the subjects were given a fourth model D, depicted in Figure 3 B, which was identical to the bas-relief model B except this time etched with grid lines. The subjects were told that the grid lines represent the scale of the model, and that scale varied with depth, such that objects in the distance were presented at a smaller scale as compared with objects in the foreground, as indicated by the converging grid lines. When offered this alternative, all the subjects chose this one, because this model embodies the same duality in size perception observed in phenomenal perspective, whereby objects in the distance are perceived to be smaller by perspective, and yet at the same time they are perceived to be undiminished in size.

Perspective is traditionally defined as a projection from a three-dimensional world onto a two-dimensional surface, such as the retina, or the film plane of a camera. But the world of experience is not a flat two-dimensional projection, it is a solid volumetric world that exhibits a conformal distortion. Nowhere in the objective world of external reality is there anything remotely resembling perspective as we observe it in phenomenal experience. The prominent violation of Euclidean geometry evident in phenomenal perspective is perhaps the clearest evidence for the world of experience as an internal rather than an external entity, for the curvature of perceived space is clearly not a property of the world itself, only of our perceptual representation of it.

What does it mean for a space to be curved? If it is the space itself which is curved, rather than just the objects within that space, then it is the definition of straightness itself which is curved in that space. In other words if the space were filled with a set of grid-lines marking straight lines with uniform spacing, as shown in Figure 2 B, those lines themselves would
be curved rather than straight, as they are in Euclidean space. However the curvature would not be apparent to creatures who live in that curved space, because the curves that are followed by those grid lines are the very definition of straightness in that space. In other words a curved object in that curved space would be defined as perfectly straight, as long as the curvature of the object exactly matched the curvature of the space it was in. If you are having difficulty picturing this paradoxical concept, and suspect that it embodies a contradiction in terms, just look at phenomenal perspective which has exactly that paradoxical property. For phenomenal perspective embodies exactly that same contradiction in terms, with parallel lines meeting at two points while passing to either side of the percipient, and while being perceived to be straight and parallel and equidistant throughout their length. This absurd contradiction is clearly not a property of the physical world, which is measurably Euclidean at least at the familiar scale of our everyday environment. Therefore that curvature must be a property of perceived space, thereby confirming that perceived space is not the same as the external space of which it is an imperfect replica.

In fact, the observed warping of perceived space is exactly the property that allows a finite representational space to encode an infinite external space. This property is achieved by using a variable representational scale, i.e. the ratio of the physical distance in the perceptual representation relative to the distance in external space that it represents. This scale is observed to vary as a function of distance from the center of our perceived world, such that objects close to the body are encoded at a larger representational scale than objects in the distance, and beyond a certain limiting distance the representational scale, at least in the depth dimension, falls to zero, i.e. objects beyond a certain distance lose all perceptual depth. This is seen for example where the sun and moon and distant mountains appear as if cut out of paper and pasted against the dome of the sky.

3. Hestenes' Projective and Conformal Mappings

David Hestenes' projective and conformal mappings provide an elegant solution to the problem of infinity by simple projection. This requires the addition of one supernumerary dimension to provide an offset vantage point from which to view Euclidean space. For three-dimensional Euclidean space this requires the addition of a fourth dimension, which is difficult to imagine, and therefore the concept is usually depicted for two-dimensional Euclidean space.

Figure 4 A shows the two-dimensional Euclidean plane viewed from a third-dimensional vantage point that transforms every point along the x axis, for example, to an angle from the vertical, with the singular property that an angle of 90 degrees projects all the way to infinity. These angles are then projected onto a two-dimensional surface where they project an image that is very much like a perspective view, or photograph of the plane, with the
Figure 4. A: A perspective projection of the two-dimensional Euclidean plane from a viewpoint in the third dimension, projected onto an image plane. B: The result of a projection from the three-dimensional Euclidean world through a fourth dimension (not shown) results in a warped three-dimensional space that maps infinity to a finite distance. C: Two such three-dimensional projections taken in different directions are incommensurable with each other, they cannot be merged seamlessly.

extraordinary property that infinity projects to a finite line that now represents “infinity”. This projection is reminiscent of the optical projection of the eye, whereby light from a three-dimensional world is projected onto a two-dimensional retina. Although the concept is most clearly illustrated in the two-dimensional case, the same principle holds also for a three-dimensional world using a supernumerary fourth-dimension as shown in Figure 4 B which shows only the three-dimensional projection not the fourth-dimensional vantage point. Although this figure may look like a regular model of a room, it actually represents an “Ames Room” distorted three-dimensional perspective in which the parallel lines of the room converge to a point at “infinity” which is not actually at infinity, but right there where the “parallel” lines meet. This three-dimensional projectively warped model is like a diorama, or a theatre set, where objects in the distance get smaller by perspective, and yet at the same time they are perceived to be undiminished in size, and the backplane painting represents the dome of the sky at “infinity”. This perspective-distorted model is very similar to our perceptual experience of
space, and demonstrates how an explicit three-dimensional model of finite size can represent an infinite external space.

There is however a fatal flaw in this concept of projection, because it can only project to space in one general direction, it cannot turn around and look at the world in the opposite direction. Two such perspectives placed back-to-back to look in opposite directions are incommensurable, they cannot be joined seamlessly, as suggested in Figure 4 C.

![Figure 5. A: Replacing the projection plane with a projection sphere allows the projection to “look” in all directions simultaneously. B: Moving the perspective vantage point to the zenith of the sphere confers some extraordinary invariances to the projection.](image)

The solution is to project not to a plane surface, but to a spherical surface that can essentially “look in all directions” as shown in Figure 5 A. However the viewpoint should not be placed at the center of the projection sphere. In the first place, it could only ever use the lower hemisphere, the upper hemisphere would remain unused. Instead, the viewpoint is located at the apex of the projection sphere as shown in Figure 5 B, which defines a stereographic projection, and the reason is because this placement confers to the projection some extraordinary invariances. The apex of the sphere now represents “infinity” in all directions all at a single singular point. Furthermore, this conformal mapping projects circles in Euclidean space to circles in conformal projection as shown in Figure 6 A (from Perwass & Hildenbrand [8]). Even more remarkably, the conformal projection projects straight lines to circles too, with the extraordinary property that all such circles pass through “infinity” at the pole of the sphere. Figure 6 B (from Perwass & Hildenbrand [8]) shows two lines on the Euclidean plane projecting to two circles in conformal projection, and those circles cross at two points: once
corresponding to their intersection on the Euclidean plane, and again where they meet at “infinity”. All straight lines in Euclidean space project to circles in conformal projection that pass through the same “infinity”.

Figure 6. A: Circles in Euclidean space project to circles in conformal space. Straight lines also project to circles that pass through “infinity” at the zenith, i.e. circles and straight lines are the “same shape” in this space. B: Two straight lines that cross in Euclidean space project to two circles in conformal space that cross at two points: Once corresponding to their intersection in Euclidean space, and again where they cross at “infinity”. C: A way of visualizing the conformal projection of a three-dimensional Euclidean space into a fourth dimension that is simultaneously orthogonal to x, y, and z.

The principle of the conformal projection is usually demonstrated for the two-dimensional Euclidean case because of the difficulty of depicting a fourth spatial dimension. However in order to serve as a model of visual perception the projection should actually be performed in four dimensions projecting back to three. Although this is difficult to visualize, there is one feature of the conformal projection that makes it possible to present an intuitive image of the three-dimensional conformal projection. In Figure 6 A and B, the z dimension serves as the viewpoint for the x,y plane because the z dimension is orthogonal to all possible directions from the origin within the x,y plane. Every possible direction from the origin around the x,y plane
points to its own unique infinity in that direction, an infinity of infinities around the infinite horizon of the Euclidean plane. Every one of those infinities maps to the self-same point in conformal projection, an infinity of infinities collapsed back to a single singular point! What this is in effect is a reversal, or swapping of proximal and distal! Every point to distal infinity maps to the singular proximal point at the zenith of the sphere. This extraordinary compactification of space allows for an intuitive depiction of the same process extended through a fourth dimension of a three-dimensional space. What we need is a direction that is simultaneously orthogonal to x, y, and z. Thanks to the extraordinary compactification of the conformal mapping, now every direction from the origin in all three dimensions points to its own unique infinity in that direction, and yet all of these infinities of infinities of infinities all map to a single singular point in the conformal mapping, which can now be depicted at the center of the sphere. This can be depicted as in Figure 6 C, where the center of the conformal sphere represents “infinity”, and the entirety of three-dimensional space out to infinity in all directions is compacted into a finite spherical volume. While this is a kind of “cheat” because we are actually using the original x,y,z dimensions to depict a fourth dimension, depicting a fourth dimension within a three-dimensional sphere, it does not matter as long as the compactified “world” within the sphere is kept isolated from external three-dimensional space. In other words it would be perfectly feasible to devise an analogical spatial mechanism that represents this “fourth” dimension within a three-dimensional sphere, where the isolation from external dimensions defines what is in effect a set of directions that are all orthogonal to x, y, and z. I propose that visual perception employs exactly this kind of isolated spherical representation in the brain as an analogical spatial model of external space.

Figure 7. A: A finite cylinder and its conformal projection as B: the cylinder grows to infinite length. C: The conformal reflection of a square base with four cylindrical columns.

Figure 7 demonstrates how this kind of isolated conformal projection would represent simple geometrical shapes. Figure 7 A shows a three-dimensional
cylinder outside the conformal sphere, and its three-dimensional stereographic projection within the conformal sphere. Figure 7 B shows a cylinder of infinite length, and its circular conformal reflection. Figure 7 C shows four pillars on a square base, and its conformal reflection within the inversive sphere.

Figure 8. A stereographic projection of a scene achieved with PhotoShop [4].

Figure 8 shows a simulation of a stereographic projection of a three-dimensional scene using Photoshop [4]. Although of course this is a two-dimensional image, it gives an impression of a three-dimensional stereographic projection of a scene from a vantage point at the center of the scene. The unit circle in the picture represents the stereographic sphere in cross section, outside the circle is the three-dimensional Euclidean space of this scene (although warped here by a fish-eye perspective that maps infinity to the periphery), while within the circle is the stereographic projection of that world. Note how the tall tower in the 2 o’clock direction is reflected across the unit sphere to appear upside-down within the conformal sphere also in the 2 o’clock direction, but whereas in external space upward is “outward” away from the center, in the conformal world within the conformal sphere
upward is “inward” toward the center of the sphere. See also how the col-
lonade through the 9 o’clock direction is reflected in the conformal sphere
wherein upward is now “inward”. Note the extraordinary compactification of
this projection whereby all of external space out to infinity in all directions is
mapped to a finite volume with the point at the center representing infinity
in all directions simultaneously in the external world.

4. Bubble World v.s. Conformal World

I spent several months pondering the relation between David Hestenes’
conformal mapping and the Bubble World distortion apparent in visual per-
ception. In some ways these two mappings appear very similar. They both ex-
hibit an extreme stereographic warping that maps infinity to a finite bounded
range. But in another sense they appear as polar opposites: the Bubble World
is large at the center and shrinks to zero at the periphery, whereas Hestenes’
conformal mapping is large in the periphery and shrinks to zero at the center.

I finally realized how these two spaces are related: One is the conformal
reflection of the other, reflected through the surface of the unit sphere as
shown in Figure 10. The largest house in the Bubble World closest to the
periphery outside the unit sphere is a reflection of the largest house in the
conformal reflection within the unit sphere, while the rest of the houses pro-
gressively shrunken by perspective as they recede outward toward infinity at
the periphery of the Bubble World are reflected in the smaller houses within
the conformal sphere progressing inwards toward “infinity” at the center of
the conformal sphere.

Figure 9. A: Comparison of the Bubble World perspective
with Hestenes’ Conformal Mapping. The Bubble World is
large in the center and shrinks to nothing at the periphery...
B: The conformal mapping is large at the periphery and
shrinks to nothing at the center.
There is a direct correspondence between this double conformal mapping and the stereographic projection of the number line depicted in Figure 1 B. Every x on the number line from one all the way to “infinity” has a corresponding reciprocal point 1/x within the unit circle. This relation between radial distance and its reciprocal holds for every direction from the origin in three-dimensional space. Indeed in the conventional view of the number line depicted in Figure 1 A the inverse function 1/x is a kind of impossibility, because it suggests a one-to-one mapping between points in the infinite range from one to infinity, and corresponding reciprocal points in the finite range from zero to one. Mathematically this is generally considered as not a problem because the reciprocal of the singular point at the origin, infinity, is
simply disallowed, and thus the paradox is circumvented. But there is something strange about the notion that however large a legitimate number can be expressed on the conventional number line, whether a physical number like the number of electrons in the universe, or a theoretical number like a googolplex, so long as the number can be given an explicit place on the conventional number line, it necessarily remains infinitely short of true infinity. So the entire infinite span from the largest conceivable number out to true infinity, finds its reciprocal at the singular point which is zero, a point-for-point mapping between an infinite range and a singular infinitesimal point. All this absurdity can be avoided more simply by abandoning the pretense that our mathematics spans all the way out to infinity, and to admit from the outset that beyond a certain range (which varies depending on one’s needs) numbers are so large that for all practical purposes they are infinitely large, a subtle but profound distinction between true infinity and merely “infinity”, a number which is now as close to our grasp as is the dome of the night sky. Indeed the symmetry of the reciprocal function, the guarantee that every number has its reciprocal, and that the reciprocal of a reciprocal is the original number, are properties that are essential to provide closure to multiplication, allowing movement of terms back and forth across the equals sign in algebraic manipulations. The reciprocal function is totally paradoxical except in the context of the conformal number circle where numbers and their reciprocals exhibit simple symmetry across the number line, as seen in Figure 1 A.

5. Modification of Hestenes’ Conformal Model

The conformal mapping between the Bubble World and its inversive reflection through the surface of the unit sphere finally made sense in my mind of the conformal mapping as proposed by David Hestenes. According to Hestenes, the conformal mapping occurs in two stages, first a stereographic projection from the Euclidean world into the unit sphere, where the special properties of that projection allow for the detection of regularities that are not plainly evident in the Euclidean world. After that, a second outward reflection that projects the results of the symmetries detected in the inversive sphere back from the inside-out inverse world out to the right-side-in Euclidean world with an inverse of the inverse transformation.

For example a three-dimensional line in external space projects to a curved arc in conformal space as shown in Figure 11 A, such that the radial distance of every point in that reflection from the origin is the inverse \((1/r)\) to the radius \(r\) of the corresponding point in the line itself. The regularity, or collinearity of the line is recognizable in the inversive reflection by the fact that the curved line is a circular arc that is part of a circle that passes through “infinity” at the center of the inversive space, as shown in Figure 11 B.
**Figure 11.** A: A line-segment in three-dimensional space projects to a circular arc in conformal space. B: Regularization processes within the conformal sphere complete the regularity of the arc segment as a finite portion of an infinite line, represented by a complete circle through “infinity”.

**Figure 12.** A: Second stage in Hestenes’ conformal mapping projects the regularities detected in the conformal space back out into external space. B: The part of the line from the center of the conformal sphere must be projected to infinity in external space.

The implications of this detected regularity are projected back out again by an inverse of the inversive function $r^2$ (every radial distance in the reflection is squared) as shown in Figure 12 A, because the square function cancels the inverse function to restore the original distance $r$ in Euclidean space.
This outward projection is not particularly useful for restoring the line itself, if the line was given in the first place. The restoration is more useful for restoring the geometrical regularities detected in the conformal reflection, for example the fact that the given line segment is a part of an infinite line that stretches to infinity in opposite directions. That extrapolation is inverse-projected from the circle in the inverse world back out into Euclidean space, where it completes the symmetry of the line out to infinity as suggested in Figure 12 B.

Now this projection stage of Hestenes’ conformal mapping bothered me from the outset, because points from the center of the inversive sphere would have to be projected all the way out to infinity. But nothing can actually project to infinity! Infinity is fine as a mathematical abstraction, but for those of us who believe that mathematics is a physical mechanism, or analog computational projection taking place within the physical brain, the notion of projection to infinity is a physical impossibility.

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v = 2 \tan(\frac{1}{2 \chi})
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**Figure 13.** Modification of Hestene’s conformal projection that projects back out to another conformal space that avoids the need for a projection to infinity.

The conformal distortion manifest in the Bubble World perspective suggests that the outward projection need not be a projection to actual infinity, but just to another conformal reflection bounded by a finite limit representing infinity, a *double conformal* projection, once inward to a finite inverse world,
then back outward again to a finite Bubble World, which uses a vergence measure of depth $v(x) = 2 \tan(1/2x)$. Actually this is none other than a one-dimensional stereographic projection from the number line to the unit circle, wrapping the infinite linear range from one to infinity into the finite bounded angular range of 0 to $\pi$ in each direction. In fact, bounding the outward projection to a finite scope solves a number of thorny problems, for example the impossibility of the reciprocal function. The inverse function is no longer a paradoxical impossibility, but is simply a point-for-point mapping from one conformal world to another.

A mathematics that pretends to encompass infinity, bears a permanent scar of profound paradox right at the point where it (supposedly) makes contact with infinity. A mathematics that acknowledges the profound impossibility of infinity, and thus places it in effigy at a distance that is less than infinite, marries with “infinity” as a spatial continuum, a structure that can in principle be implemented in a finite physical mechanism like the human brain.

6. A Perceptual Model

The peculiar inverse relationship between Bubble World and conformal spaces suggests a perceptual model in which the inverse conformal reflection of a sensory input serves to detect regularities hidden in a two-dimensional stimulus, and uses those regularities to project a three-dimensional image of the objects in the world most likely to have been the cause of that stimulus. In other words, the conformal model helps to solve the inverse optics problem that seeks to invert the projection of the eye to reconstruct a three-dimensional world consistent with the two-dimensional retinal image, as suggested with a simple example in Figure 14.

A line in external space is projected to a two-dimensional retinal projection by the optics of the eye, represented by the rectangular patch on the spherical surface of the conformal sphere in Figure 14 A. The information of three-dimensional depth is lost in the optical projection. The retinal image is then inverse-projected into the inverse conformal sphere where it spreads throughout a planar probability field that represents all of the possible locations in depth that project to that same linear stimulus, suggested by the gray shading in Figure 14 B. This expresses the fundamental ambiguity in monocular vision whereby the depth value is lost in the optical projection. The inverse projection represents a simultaneous reification or reconstruction of every possible edge at every possible orientation that all project to that linear retinal stimulus. From that infinite set of alternative interpretations, the visual system selects one that has the greatest simplicity, or symmetry, (Gestalt prägnanz) in the conformal reflection. For example one possible interpretation of the linear stimulus is as a straight line that stretches to infinity in opposite directions, recognized by the fact that its inverse projection in the
Figure 14. A: A line-segment in external space projects to a retinal image which is projected onto the outer surface of the conformal sphere. B: The linear stimulus is inverse-projected inward to define a spatial probability field representing the locus of all possible lines that project to the given stimulus.

conformal sphere defines a full circle passing through the origin, as suggested in Figure 15 A.

Figure 15. A: Regularization processes detect and complete any symmetries found in the stimulus, for example completing the circular arc to a full circle passing through “infinity”. B: The implications of the detected regularity are projected back out again to a conformally warped Bubble World perspective.
The result of that regularity detection process is then projected back out, not into the external visual world that was the original source of the visual stimulus as in Hestenes’ conformal projection, but out into a conformal Bubble World representation of that external space wrapped up in a finite spherical representation whose outer surface represents infinity in all directions. It is noteworthy that the external Euclidean world, the conformally warped Bubble World, and the bizarre inverse conformal world, all project radially to the same retinal image.

7. Non-Euclidean Geometries

The peculiar warp observed in the Bubble World perspective and its inversive counterpart in the conformal sphere is reminiscent of non-Euclidean geometries. Three mathematicians, Karl Friedrich Gauss, Nicolai Lobachevsky, and János Bolyai, each independently wondered while studying Euclid’s “Elements” why Euclid had not bothered to prove his “Fifth Postulate” with the same rigor as he had the rest of his postulates. Essentially the Fifth Postulate states that if two lines that cross a third line form internal angles that sum to less than 180 degrees, then those lines must cross somewhere. This is equivalent to the “Parallel Postulate” that parallel lines never meet, and it is also equivalent to the rule that the internal angles of a triangle sum to 180 degrees. Each of those three mathematicians set out to prove the Fifth Postulate, and all three of them failed, because although the postulate seems self-evident, it is in fact impossible to prove.

That in turn opened the possibility for non-Euclidean geometries, i.e. it is possible to define a whole non-linear equivalent to Euclidean geometry that works in a space with positive curvature like the Bubble World perspective. And likewise by symmetry, one that works in a space with negative curvature, like the inversive conformal world in Hestenes’ conformal projection. For example the Pythagorean theorem can be demonstrated and proven in the distorted non-Euclidean spaces just as easily as it can in Euclidean form as suggested in Figure 16. Euclidean geometry is a subset of a whole family of non-Euclidean geometries through a range of curvatures both positive and negative.

Gauss was genuinely disturbed that we cannot be sure that our world is truly Euclidean, it could just as well be of a non-Euclidean geometry. This perceptual model suggests that Gauss’ fear was well founded, that our Bubble World perspective that we see in the world around us does indeed exhibit a non-Euclidean geometry with positive curvature, and the inversive conformal world suggested by Geometric Algebra is a mirror image of that world in a space with negative curvature. The two nonlinear geometries are connected to each other and to external reality by the Euclidean geometry that they share in common.
The Pythagorean theorem can be proven not only in Euclidean space, but also in a nonlinear space with positive curvature, like the Bubble World, and also in a nonlinear space with negative curvature, as in the conformal space.

8. Conclusion

The biological theory of the origins of mathematics offers a novel motivation for studying the laws of mathematics as evidence for the computational principles of cognitive thought, inherited directly from the properties of perception. One of the most prominent features of visual perception is the fact that the perceived world is an explicit spatial structure. This fact is usually overlooked because the world that we see around us in visual experience is all too easily confused with the real world. The prominent warp observed in phenomenal perspective, and the fact that visual experience has a variable representational scale, are clear evidence that the world of experience is not the infinite external world itself, but a finite perceptual replica of that world.
in an internal representation. The Bubble World perspective suggests that the reason for that warp is to allow a finite spatial model, like the world we see around us in visual experience, to model a practically infinite external world.

Hestenes' conformal projection highlights a similar representational issue in mathematics with the treatment of infinity, a paradoxical anomaly at the ends of the number line. As in perception, infinity can be projected back to “finity” through the conformal mapping, seen in simplest form in the stereographic projection of the number line shown in Figure 1 A onto the number circle. The resemblance to the Bubble World perspective in perception suggests that this stereographically warped number circle is a more direct or primal representation of the concept of number in the human mind than the conventional number line, as seen in the ease with which in colloquial, or “naive math” we discuss “infinity” as if it were a single place located on the number line, without regard to the inconceivable vastness of that concept, relative to which even the greatest finite magnitudes shrink to infinitesimal nothing by comparison.

One advantage of this very peculiar representation of number is that it resolves the mathematical paradox of the reciprocal function, the supposed one-to-one mapping between every point in the infinite open range from one to infinity to its reciprocal within the finite bounded range from zero to one. While this concept is perfectly permissible as a mathematical abstraction, it is not the kind of thing that can be implemented in an explicit analogical representation because an infinite range cannot map point-for-point with a finite bounded range. In the conformally warped number line that paradoxical mapping loses its paradox and becomes a simple one-to-one mapping between numerical values from one to infinity found on the upper semicircle of the number circle, and their corresponding reciprocals between zero and one on the lower semicircle. The number and its reciprocal exhibit a mirror symmetry across the horizontal number line. Not only does this provide complete closure providing every numerical value including zero with a legitimate reciprocal, but the loss of resolution due to the stereographic projection with approach toward “finity” at the zenith of the number circle is mirrored by the same point-for-point loss of resolution in the lower semicircle with approach toward the origin, i.e. there are as many “integers per degree of arc” numerical resolution approaching the zenith of the number circle as there are for the reciprocals of each of those integers approaching the nadir of the number circle near zero.

A similar principle obtains for the full three-dimensional conformal mapping proposed by Hestenes. The relationship between a point in three-dimensional space and its conformal reflection within the conformal sphere is exactly like the relation between a point on the number line outside of unity and its reciprocal within the bounded unitary range. Like the reciprocal function, Hestenes' conformal mapping maps the three-dimensional space beyond
unit distance out to infinity in all directions to its conformally mapped spatial reciprocal located within the finite bounded volume of the unit sphere. Here again is a paradoxical point-for-point mapping from an infinite surrounding space to a finite bounded sphere. The second stage of Hestenes' conformal projection requires a projection outward potentially to infinity in a manner that defies explicit implementation in any kind of analogical computational mechanism. The solution to this paradox can be found as in the stereographic number circle, which is to extend the conformal mapping outside the unit sphere, out into a larger surrounding sphere whose outer surface represents “infinity” in all directions. This now establishes a true point-for-point mapping between a point in the outer conformal sphere and its spatial reciprocal within the inner inverse conformal sphere exactly analogous to the relation between a number and its reciprocal on the number circle. There are as many integers per unit distance numerical resolution (or grid lines per milimeter spatial resolution of Figure 10) in the compressed scale near the outer boundary of the outer conformal sphere as there are reciprocals per unit distance (corresponding grid lines) of each of those integers within the inner inverse conformal sphere in the compressed scale near the origin. Unlike Hestenes projection out to infinity, this is a representational scheme that can be implemented in an explicit analogical computational mechanism.

Like most of mathematics, Hestenes' conformal mapping serves to detect and complete regularities in the real world which is basically Euclidean (at least at the normal scale of everyday human interaction) and that is why the regularities detected within the conformal world are projected back out into Euclidean space where the regularities that were found in the conformal space can be matched against external reality. The Double Conformal model on the other hand does not seek to model the real external world, but instead it seeks to model the perspective warped perceptual world in the human mind, motivated by the biological theory of the origins of mathematics.

The fact that the internal and external conformal worlds both project the same image to the spherical surface by which they are connected, and the fact that that image also matches the optical projection of the corresponding Euclidean scene, strongly suggests a perceptual model whereby the inner and outer conformal models serve to interpret the two-dimensional retinal image simultaneously in the inner and outer conformally mapped worlds in order to reconstruct the spatial configuration of external reality as an explicit spatial structure in the brain that is most likely responsible for the given retinal projection. While the conformal mapping proposed by Hestenes occurs in two discrete passes, first inward to the inversive conformal model and then back outward to the external world, the conformal mirror symmetry of the two concentric spaces in the Double Conformal model suggest a continuous emergent process whereby features detected in the internal inversive conformal sphere are projected back out to the outer Bubble World conformal sphere, while at the same time features detected in the Bubble World sphere are projected back inward to the inversive conformal sphere, so that
the final emergent percept makes use of regularities detected in either of the two conformal worlds, moving back and forth between them like algebraic manipulations moving terms freely back and forth across the equals sign, but as a continuous emergent process.

The explicit spatial nature of algebra revealed by Geometric Algebra is consistent with algebra as explicit spatial computations in a spatial medium. This suggests that the explicit spatial manipulations of vector algebra are more basic and primal than the scalar algebra that we first learn in school, and that scalar algebra is an abstraction of the spatial geometric algebra and reflects its spatial logic in abstract form. Geometric Algebra has been characterized as a reflection algebra because its most primal computation, vector multiplication, can be reduced to a simple reflection. Even the conformal mapping can be described by a conformal reflection as through a curved mirror, and that reflection corresponds to a multiplicative inverse between the inner and outer conformal worlds. Projections are also primal to Geometric Algebra, another explicitly spatial computational principle, whether projecting “outward” as in the wedge product, or “inward” as in the dot product. The fact that the fantastically elaborate manipulations of Geometric Algebra appear to be composed of simple projections and reflections suggests a wave-based principle of representation in the brain whereby spatial computations are performed by explicit spatial reflections and projections between cyclic standing waves in the brain.

References

[5] Lehar S. (1999)Hallway Experiment. An informal experiment conducted in a hallway at the Schepens Eye Research Institute, (The outcome is so “obvious” that no formal experiment was really necessary, the reader can confirm the result for themself in any convenient hallway) documented on-line at http://cns-alumni.bu.edu/~slehar/hallway/hallway.html
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